and lower envelopes of a body with an initial attitude and rate.

Using the data of Fig. 2 in Eq. (14) and (15), predictions of the upper and lower bounds of angle of attack for a body with trim and initial attitude and rate are shown in Fig. 3 along with 6 degree-of-freedom results. The upper bound of Fig. 3 was computed using the sum of the two maximums from Fig. 2. The lower bound of Fig. 3 was computed using the envelope values from Fig. 2 which resulted in the smallest magnitude. The comparison Fig. 3, shows that the sum of individual contributions bound the 6 degree-of-freedom computer results.

The analysis indicates that the individual effects of trim and initial conditions can be combined in a straightforward manner for a range of conditions including a variable trim history to obtain bounds on the envelope of angle of attack.

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Derivation of the Shell Compatibility Equations

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Introduction

THE strain compatibility equations have been treated in numerous works on thin shell theory. The first attempt to derive the compatibility equations in shell theory is attributed to Odqvist in 1937.[1]

The first complete derivation of the strain compatibility equations for linear shell theory was given by Gol'denveizer.[2] In his work, Gol'denveizer utilized the condition that the shell displacements must be independent of the path in shell space when calculated from the linearized strain-displacement relations.

Derivations of the compatibility equations in linear and nonlinear shell theory also have utilized the Gauss and Mainardi-Codazzi conditions as a basis.[3–5] These geometric conditions can be written in terms of the membrane and bending strains. Satisfying the resulting set of differential equations insures that the strain measures are analytic functions of the shell reference surface displacements.

The purpose of this Note is to re-examine the shell strain compatibility equations from a physical aspect. The derivation is based on the requirement that the shell membrane and bending strains must be independent of the natural paths from point to point on the reference surface. The natural paths are taken to the shell reference surface coordinates.

The derivation given in this paper is restricted to a discussion of the strain compatibility conditions on an arbitrary reference surface in the shell continuum, thus implying the assumption of the Kirchhoff hypotheses. As a consequence, the state of strain in the surface can be described exclusively in terms of the first and second fundamental forms of deformed and undeformed surfaces. The derivation does not rely on any form of the Gauss and Mainardi-Codazzi equations as a basis. Instead, these equations arise as a natural consequence of the arguments presented herein.

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Derivation of Compatibility Equations

The reference surface S of a shell is described by curvilinear coordinates ξ^α ($\alpha = 1, 2$). As a result of external forces, the shell reference surface deforms to S^*. The relationship between S and S^* can be conveniently described in terms of the displacements, i.e.,

$$X = x^i + \phi^i \tag{1}$$

In what follows, English indices extend over the range 1, 2, 3, while Greek indices extend over the range 1, 2. Equation (1) describes the position of a point Q^* on S^* in terms of the position vector x^i of a point Q on S and the displacements ϕ^i. The displacement vector ϕ^i and position vector x^i are functions of the coordinates ξ^α.

The vectors tangent and normal to the coordinate curves at point Q^* and S^* are given, respectively, by

$$dX^i = X_\alpha{}^i d\xi^\alpha \tag{2}$$

$$N_i = \tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon_{ijk}X_\alpha{}^j X_\beta{}^k \tag{3}$$

The quantities, $\epsilon^{\alpha\beta}$ and ϵ_{ijk}, in Eq. (3), are the permutation symbols of the deformed and undeformed surface, respectively.[6] The coefficients of the first and second fundamental tensors of the surface S^*, respectively, are defined as

$$G_{\alpha\beta} = X_\alpha{}^i X_\beta{}^i \tag{4a}$$

$$B_{\alpha\beta} = X^i{}_{\alpha|\beta}N^i = -X_\alpha{}^i N^i{}_{|\beta} \tag{4b}$$

The vertical bar in Eq. (4) denotes covariant differentiation with respect to the metric of the surface S^*. Equations (4a) and (4b) have been used as measures of the stretching and bending of the shell reference surface, respectively. Thus

$$2e_{\alpha\beta} = G_{\alpha\beta} - g_{\alpha\beta} \tag{5a}$$

$$\rho_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} \tag{5b}$$

where $g_{\alpha\beta}$ and $b_{\alpha\beta}$ are the first and second fundamental forms for the undeformed reference surfaces.

The base vectors at point Q^* on S^* are considered here to be functions of ξ^α belonging to class C^2, and it is desired to express the base vectors at point T^* on S^* in terms of the base vectors at Q^*. The coordinates of the shell surface makes it possible to choose two obvious paths from point Q^* to point T^*, as shown in Fig. 1. The base vectors at T^* that result from following path $Q^* - U^* - T^*$ are denoted by $\tilde{X}_\alpha{}^i$ and $\tilde{N}_i$. Those base vectors at T^* resulting from following path $Q^* - V^* - T^*$ are denoted by $\hat{X}_\alpha{}^i$ and $\hat{N}_i$. The membrane and bending strains at T^* for the two sets of base

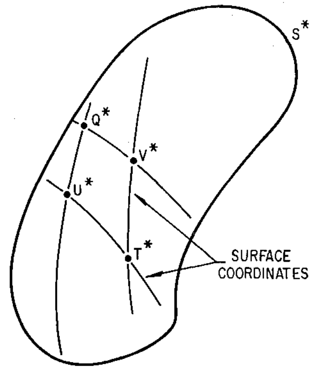

Fig. 1 Natural paths on the deformed surface.

vectors, according to Eq. (5), are

$$2\ \tilde{e}_{\alpha\beta} = \tilde{G}_{\alpha\beta} - g_{\alpha\beta} \quad (6a)$$

(at T^* resulting from path $Q^* - U^* - T^*$)

$$\tilde{\rho}_{\alpha\beta} = \tilde{B}_{\alpha\beta} - b_{\alpha\beta} \quad (6b)$$

and

$$2\ \hat{e}_{\alpha\beta} = \hat{G}_{\alpha\beta} - g_{\alpha\beta} \quad (7a)$$

(at T^* resulting from path $Q^* - V^* - T^*$)

$$\hat{\rho}_{\alpha\beta} = \hat{B}_{\alpha\beta} - b_{\alpha\beta} \quad (7b)$$

The membrane and bending strains at point T^* should be equivalent regardless of the path from Q^*. Thus, equating the membrane and bending strains of Eqs. (6) and (7),

$$\tilde{e}_{\alpha\beta} = \hat{e}_{\alpha\beta} \quad (8)$$

$$\tilde{\rho}_{\alpha\beta} = \hat{\rho}_{\alpha\beta} \quad (9)$$

By using Eq. (4) and the appropriate base vectors, Eqs. (8) and (9) become

$$\tilde{X}_{\alpha}{}^{i}\tilde{X}_{\beta}{}^{i} = \hat{X}_{\alpha}{}^{i}\hat{X}_{\beta}{}^{i} \quad (10)$$

$$\tilde{X}^{i}{}_{\alpha|\beta}\tilde{N}^{i} = \hat{X}_{\alpha|\beta}{}^{i}\hat{N}^{i} \quad (11)$$

Adding $-\tilde{X}_{\alpha}{}^{i}\hat{X}_{\beta}{}^{i}$ to both sides of Eq. (10) and rearranging,

$$(\tilde{X}_{\alpha}{}^{i} + \hat{X}_{\alpha}{}^{i})(\tilde{X}_{\beta}{}^{i} - \hat{X}_{\beta}{}^{i}) = 0 \quad (12)$$

Equation (12) implies

$$\tilde{X}_{\beta}{}^{i} = \hat{X}_{\beta}{}^{i} \quad (13)$$

Thus, Eq. (11) reduces to

$$\tilde{N}_{i} = \hat{N}_{i} \quad (14)$$

Equations (13) and (14) are statements of the strain compatibility conditions in a broad sense. It is necessary, however, to determine the meaning of these equations in terms of the base vectors at point Q^* on S^*. To accomplish this, the base vectors $\tilde{N}_i$, $\tilde{X}_{\alpha}{}^{i}$ and $\hat{N}_i$, $\hat{X}_{\alpha}{}^{i}$ will be written in terms of the base vectors at Q^* by means of a second-order Taylor's expansion. The expansion of the base vectors at Q^* will be made along one surface coordinate curve, ξ^{ρ}, while ξ^{λ} is constant ($\xi^{\rho} \neq \xi^{\lambda}$). Then, an expansion will be made along the surface coordinate curve ξ^{λ}, while ξ^{ρ} is held constant ($\xi^{\rho} \neq \xi^{\lambda}$). This process would correspond to the path $Q^* - U^* - T^*$ shown in Fig. 1. The path $Q^* - V^* - T^*$ on S^* would involve a Taylor's expansion along ξ^{λ}, holding $\xi^{\rho} =$ const. ($\xi^{\lambda} \neq \xi^{\rho}$); then along ξ^{ρ}, holding ξ^{λ} constant. Thus, the two sets of base vectors at T^* are given by

$$\tilde{X}^{i}{}_{\alpha} = X^{i}{}_{\alpha} + X^{i}{}_{\alpha|\rho}d\xi^{\rho} + X^{i}{}_{\alpha|\lambda}d\xi^{\lambda} + \tfrac{1}{2!}X^{i}{}_{\alpha|\lambda\lambda}(d\xi^{\lambda})^2 + X^{i}{}_{\alpha|\lambda\rho}d\xi^{\lambda}d\xi^{\rho} + \tfrac{1}{2!}X_{\alpha|\rho\rho}{}^{i}(d\xi^{\rho})^2 \quad (15)$$

$$\hat{X}^{i}{}_{\alpha} = X^{i}{}_{\alpha} + X^{i}{}_{\alpha|\lambda}d\xi^{\lambda} + X^{i}{}_{\alpha|\rho}d\xi^{\rho} + \tfrac{1}{2!}X^{i}{}_{\alpha|\rho\rho}(d\xi^{\rho})^2 + X^{i}{}_{\alpha|\rho\lambda}d\xi^{\rho}d\xi^{\lambda} + \tfrac{1}{2!}X_{\alpha|\lambda\lambda}{}^{i}(d\xi^{\lambda})^2 \quad (16)$$

$$\tilde{N}_{i} = N_{i} + N_{i|\lambda}d\xi^{\lambda} + N_{i|\rho}d\xi^{\rho} + \tfrac{1}{2!}N_{i|\rho\rho}(d\xi^{\rho})^2 + N_{i|\lambda\rho}d\xi^{\lambda}d\xi^{\rho} + \tfrac{1}{2!}N_{i|\lambda\lambda}(d\xi^{\lambda})^2 \quad (17)$$

$$\hat{N}_{i} = N_{i} + N_{i|\rho}d\xi^{\rho} + N_{i|\lambda}d\xi^{\lambda} + \tfrac{1}{2!}N_{i|\lambda\lambda}(d\xi^{\lambda})^2 + N_{i|\rho\lambda}d\xi^{\rho}d\xi^{\lambda} + \tfrac{1}{2!}N_{i|\rho\rho}(d\xi^{\rho})^2 \quad (18)$$

The quantities $(\ )_{|\rho}$ and $(\ )_{|\rho\lambda}$ are first- and second-order tensor derivatives. Substituting Eqs. (15) and (16) into Eq. (13) and simplifying,

$$X^{i}{}_{\alpha|\rho\lambda} = X^{i}{}_{\alpha|\lambda\rho} \quad (19)$$

Similarly, substituting Eqs. (17) and (18) into Eq. (14), the conditions for bending strain compatibility are

$$N_{i|\rho\lambda} = N_{i|\lambda\rho} \quad (20)$$

Equations (19) and (20) are equivalent to the equations of strain compatibility given by Gol'denveizer on pp. 55–59 of Ref. 2. By using these equations, three independent relationships are obtained. These can be written in the form

$$\epsilon^{\gamma\alpha}\epsilon^{\lambda\rho}[\Gamma_{\gamma\alpha\rho,\lambda} + \Gamma^{\sigma}{}_{\gamma\rho}[\Gamma_{\sigma\alpha\lambda} + B_{\gamma\rho}B_{\alpha\lambda}] = 0 \quad (21)$$

$$\epsilon^{\lambda\rho}B_{\gamma\lambda|\rho} = 0 \quad (22)$$

where $\Gamma_{\gamma\alpha\rho}$ and $\Gamma_{\gamma\rho}{}^{\sigma}$ are the Christoffel symbols of the first and second kinds, respectively, for the deformed surface.

Equations (21) and (22) are the Gauss and Mainardi-Codazzi conditions, respectively. By using Eqs. (7) and the definitions of the Christoffel symbols, these equations can be transformed into the general nonlinear strain compatibility conditions for a shell surface. This procedure is discussed in Ref. 4 and will not be repeated here.

Concluding Remarks

By requiring the measures of the membrane and bending strains to be independent of the natural paths from point to point on the deformed reference surface, the shell compatibility conditions can be derived in a straightforward manner. This derivation has the advantage of being based upon physical arguments rather than wholly mathematical constraints as in other papers. Indeed these mathematical constraints, which have served as the basis for the strain compatibility equations in the past, are obtained directly from this derivation.

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Nucleating Mechanism for Spall in Aluminum

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SPALL in solids, defined as a complete or partial separation of material resulting from the tension induced by the interaction of two rarefaction waves can be experimentally studied using the gas gun. Gas guns such as those in Ref. 1 use a projectile to carry a flat flyer plate which impacts on a target. Recovery of the target for detailed examination becomes difficult because of the destruction or damage to the target after flyer plate-target impact. A scheme† to eliminate the damage after impact is shown in Fig. 1. The flyer plate is designed to break away from the piston at impact and pass over the sting. The aluminum foam with its low impedance will attenuate the transmitted wave and crush it to a

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